Lemma: (Pumping Lemma) If $L$ is a regular language, then there exists a positive integer $p$ (the pumping length) such that every string $s \in L$, $|s| \geq p$, can be partitioned into three pieces, $s = xyz$, such that the following conditions hold:

- $|y| > 0$,
- $|xy| \leq p$, and
- for each $i \geq 0$, $xy^i z \in L$, 

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Strategy for showing that a language $L$ is not regular

1. Assume that $L$ is regular (with the aim of reaching a contradiction).

2. Choose a string $s \in L$, such that $|s| \geq p$ where $p$ is the pumping length given by the pumping lemma.
   - The pumping lemma then says that $s$ can be partitioned into three pieces $s = xyz$ where $|y| > 0$ and $|xy| \leq p$, such that $xy^iz \in L$ for all $i \geq 0$.

3. Show that for ALL POSSIBLE partitions of $s = xyz$ satisfying $|y| > 0$ and $|xy| \leq p$ there exists an $i$ such that $xy^iz \notin L$.
   - If we succeed to show this, then we have a contradiction with the pumping lemma and our assumption that $L$ is regular is wrong.
The standard example

\[ L = \{a^n b^n \mid n \geq 0\} \text{ is not regular.} \]

Proof:

1. Assume that \( L \) is regular (with the goal of reaching a contradiction).

2. Choose \( s = a^p b^p \) where \( p \) is the pumping length given by the pumping lemma.

   - \( s \in L \) and \( |s| \geq p \), so the pumping lemma says that \( s \) can be partitioned into three pieces \( s = xyz \) where \( |y| > 0 \) and \( |xy| \leq p \), such that \( xy^iz \in L \) for all \( i \geq 0 \).

3. \( s = a^p b^p = xyz \) where \( |y| > 0 \) and \( |xy| \leq p \) implies that for all such partitions of \( s \), \( y \) is a string of \( a \)'s of length at least 1. Choose \( i = 2 \), \( xy^2z \) contains more \( a \)'s than \( b \)'s, thus, \( xy^2z \notin L \).

   - This contradicts the pumping lemma, hence, our assumption that \( L \) is regular is wrong and consequently \( L \) is not regular.
Primes represented in unary

$L = \{1^n \mid n \text{ is a prime number}\}$ is not regular.

**Proof:** Assume that $L$ is regular. Let $p$ be the pumping length for $L$ given by the pumping lemma.

Choose $s = 1^n$ where $n$ is a prime and $n > p + 1$. $s \in L$ and $|s| \geq p$, hence, $s$ can be partitioned into $xyz$ satisfying the conditions in the pumping lemma. $|xy| \leq p$ implies $|z| > 1$. Since $|z| > 1$ we have $|xz| > 1$.

Let $i = |xz|$, then $|xy^iz| = |xz| + |y||xz| = (1 + |y|)|xz|$. Since both $(1 + |y|)$ and $|xz|$ are at least 2, their product cannot be a prime. Thus, $|xy^iz|$ is not a prime. Contradiction, the assumption is wrong and $L$ is not regular.
$L = \{0^i1^j \mid i > j\}$ is not regular.

**Proof:** Assume that $L$ is regular. Let $p$ be the pumping length for $L$ given by the pumping lemma. Choose $s = 0^{p+1}1^p$. $s \in L$ and $|s| \geq p$, hence, $s$ can be partitioned into $xyz$, satisfying the conditions in the pumping lemma. The condition $|xy| \leq p$ implies that $y$ consists only of 0’s.

The pumping lemma says that $xy^iz \in L$ even if $i = 0$, $xy^0z = xz$ which gives $xz \in L$, and since $|y| > 0$ we know that $xz$ contains fewer 0’s than $xyz = s$. Since $s$ only has one more 0 than 1’s, we conclude that $xz$ does not contain more 0’s than 1’s and $xz \notin L$.

**Contradiction** (between $xz \in L$ and $xz \notin L$), the assumption must be wrong and $L$ is not regular.